ARMY RESEARCH LABORATORY

# Analytical Approach to Ballistic Dispersion of Projectile Weapons Based on Variant Launch Velocity

**by Michael M. Chen**

**ARL-RP-437** **April 2013**

**A reprint from *Journal of Applied Mechanics,* Vol. 78, pp. 031015-1–031015-9, May 2011.**

# Army Research Laboratory

Aberdeen Proving Ground, MD 21005-5066



**ARL-RP-437** **April 2013**


# Analytical Approach to Ballistic Dispersion of Projectile Weapons Based on Variant Launch Velocity


**Michael M. Chen**
**Weapons and Materials Research Directorate, ARL**



**A reprint from *Journal of Applied Mechanics,* Vol. 78, pp. 031015-1–031015-9, May 2011.**

# REPORT DOCUMENTATION PAGE

*Form Approved*
*OMB No. 0704-0188*



**1. REPORT DATE** *(DD-MM-YYYY)*
April 2013

**2. REPORT TYPE**
Reprint

**3. DATES COVERED (From - To)**
October 2009–September 2010

**4. TITLE AND SUBTITLE**
Analytical Approach to Ballistic Dispersion of Projectile Weapons Based on Variant Launch Velocity

**5a. CONTRACT NUMBER**

**5b. GRANT NUMBER**

**5c. PROGRAM ELEMENT NUMBER**

**6. AUTHOR(S)**
Michael M. Chen

**5d. PROJECT NUMBER**

**5e. TASK NUMBER**

**5f. WORK UNIT NUMBER**

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
U.S. Army Research Laboratory
ATTN: RDRL-WML-G
Aberdeen Proving Ground, MD 21005-5066

**8. PERFORMING ORGANIZATION REPORT NUMBER**
ARL-RP-437

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**

**10. SPONSOR/MONITOR'S ACRONYM(S)**

**11. SPONSOR/MONITOR'S REPORT NUMBER(S)**

**12. DISTRIBUTION/AVAILABILITY STATEMENT**
Approved for public release; distribution is unlimited.

**13. SUPPLEMENTARY NOTES**
A reprint from *Journal of Applied Mechanics,* Vol. 78, pp. 031015-1–031015-9, May 2011.

**14. ABSTRACT**
This study quantifies the contribution of variant projectile velocity at a gun muzzle to its dispersion at an aim point from an analytical approach. The dispersion was formulated on the basis of stochastic physical conditions, including potential crosswind effect. As a result, the statistical quantities of projectile impact distribution could be obtained from the stochastic formulation. In addition, this research proposes correction factors that may be needed when comparing dispersion in angular mil at multiple downrange distances. The significance of the correction factors was demonstrated through a few application examples.

**15. SUBJECT TERMS**
ballistic dispersion, small calibers, weapon accuracy, target impact points, statistical formulation

| 16. SECURITY CLASSIFICATION OF: | | | 17. LIMITATION OF ABSTRACT | 18. NUMBER OF PAGES | 19a. NAME OF RESPONSIBLE PERSON |
|---|---|---|---|---|---|
| a. REPORT | b. ABSTRACT | c. THIS PAGE | | | Michael M. Chen |
| | | | | | 19b. TELEPHONE NUMBER (*Include area code*) |
| Unclassified | Unclassified | Unclassified | UU | 14 | 410-306-0756 |

**Standard Form 298 (Rev. 8/98)**
Prescribed by ANSI Std. Z39.18

# Analytical Approach to Ballistic Dispersion of Projectile Weapons Based on Variant Launch Velocity

**Michael M. Chen**
Ballistic Structure and Launch Dynamics Branch,
Weapons and Materials Research Directorate,
U.S. Army Research Laboratory,
RDRL-WML-G,
Aberdeen Proving Ground, MD 21005
e-mail: michael.m.chen@us.army.mil

*This study quantifies the contribution of variant projectile velocity at a gun muzzle to its dispersion at an aim point from an analytical approach. The dispersion was formulated on the basis of stochastic physical conditions including potential crosswind effect. As a result, the statistical quantities of projectile impact distribution could be obtained from the stochastic formulation. In addition, this research proposes correction factors that may be needed when comparing dispersion in angular mil at multiple downrange distances. The significance of the correction factors was demonstrated through a few application examples.* [DOI: 10.1115/1.4003430]

## 1 Introduction

Ballistic dispersion is one of the important metrics that has been used to assess the performance of a weapon system. Over the past two decades, much research has been conducted to characterize target impact dispersion. In general, a wide range of ammunitions, including large-, medium-, and small-caliber projectiles, along with a few different weapon systems, including air guns and electromagnetic guns, has been covered for dispersion study [1–4]. Most investigations focused on dispersion analysis of experimental data for a particular weapon system. Some published research discussed ballistic dispersion with the consideration of a predefined physical condition from external environment, and others addressed the phenomenon with the demonstration of computer modeling and simulations [5,6].

Conventionally, the absolute value of dispersion was initially computed based on a number of rounds at a certain range. Then, the dispersion quantity was divided by the corresponding downrange distance to obtain the dispersion in angular mil. This dimensionless unit enables us to compare dispersions at two or more different downrange distances. This comparison of dispersion at multiple distances is valid under the assumption that the projectile flies in a vacuum space. In other words, no external factor is considered in the calculation. Thus, one of the objectives of this study is to formulate the random nature of round-to-round errors in consideration of several varying environmental factors and evaluate if the underlying assumption for the conventional dispersion calculation is appropriate when some external forces are taken into account during flight. This analytical approach links dispersion results to projectile initial velocity conditions or vice versa from a theoretical point of view. When one of these two measurements is available, the other quantity could be derived accordingly. The proposed formulas in this study may be utilized to facilitate performance assessments of weapon systems.

This investigation begins with the introduction of the most simplistic case, one where no external force is applied on a projectile during flight. That is, the initial conditions of the projectile at the muzzle remain the same throughout the entire flight period. It has been observed from many experiments that the target impact points (TIPs) were randomly distributed even when the same ammunition and weapon types were fired. A reasonable interpretation is that the spread of the TIP in the elevation direction is mainly driven by the variation in the initial velocity of the projectile in the elevation direction when it exits a gun barrel. In addition, the spread of the TIP in the azimuth direction can be assumed to be primarily driven by the variation in the initial velocity of the projectile in the azimuth direction. It has been known that the variation in the projectile initial velocities may be influenced by asymmetric geometry, uneven in-bore pressure, imperfect straightness of gun tubes, etc. The uncertainty from one round to another can be better modeled in terms of probability distribution functions. When a large number of rounds are considered, it has been observed that the initial velocity in each elevation and azimuth direction is approximately a Gaussian distribution. This approximation of a Gaussian random variable to the initial velocity will be used throughout the study.

High-speed projectiles, such as Mach 3 or higher, exhibit varying initial velocities in both elevation and azimuth directions at the gun muzzle. Thus, the velocities are considered to be random variables. However, the initial downrange velocity was found to possess a fairly low coefficient of variation [7] and, therefore, can be assumed to be a constant. Furthermore, three external factors—drag force, gravity drop, and crosswind—commonly considered to influence dispersion of TIP, are taken into account in the formulation of ballistic dispersion. It is well known that as the downrange velocity decreases with the increase in downrange travel distance due to air drag, the flight time is no longer linearly proportional to the travel distance. In addition, the effect of gravity will accelerate or decelerate the initial velocity in the elevation direction, depending on whether the initial velocity is going downward or upward. There are also two possible scenarios when considering crosswind, i.e., the initial lateral velocity of a projectile being of the same or opposite direction of the crosswind. With the aid of probability theory and statistics, the association of ballistic dispersion with these physical effects is demonstrated in a stochastic approach.

For low-speed projectiles, experimental data have shown that the coefficients of variation in the initial downrange velocity were significantly higher than those for high-speed projectiles. Thus, to be more accurate, the initial downrange velocity should be treated as a random variable. Some of the known causes regarding the phenomena include time-varying frictions between obturator and bore surface, gun barrel manufacturing tolerance, granular shape variations of propellant charges, packaging deviations of each propellant load, distinctive flame-spreading path, changing ambient temperature, etc. Since the contribution of these factors to the variations in initial velocities is significant in all three Cartesian directions, all the assumptions for high-speed projectiles remain effective. As a result, the initial downrange velocity is represented

Contributed by the Applied Mechanics Division of ASME for publication in the Journal of Applied Mechanics. Manuscript received November 5, 2010; final manuscript received December 23, 2010; accepted manuscript posted January 12, 2011; published online February 16, 2011. Assoc. Editor: Bo S. G. Janzon.

  

by a nonzero-mean Gaussian random variable. The mean of the random variable is the overall average of the downrange velocity for the weapon/ammunition system.

Another major goal of this research is to propose correction factors (CFs) that may be needed when comparing dispersion in angular mil at multiple downrange distances. Application examples of the CF formulation are demonstrated. In addition, the sensitivity of the CF to drag coefficient, projectile mass, air density, crosswind speed, etc., is also investigated. A parametric study is performed to gain better understanding of the influencing parameters. It is of interest to learn to what degree these parameters contribute to dispersion and what can be expected regarding the properties of ballistic dispersion. The CF is found to be significant in several cases and therefore should be taken into account when dispersion comparison is made.

Finally, it should be mentioned that with the consideration of the variation in the initial velocities in all three Cartesian components, the formulation of the dispersion appears to be quite complex but robust. This study accounts for the effects of aerodynamic drag, gravity, and crosswind. It is by no means an indication that ballistic dispersion is influenced only by these factors. For instance, the effects of yaw rate and Coriolis drift for long range projectiles are ignored. Caution shall be taken when employing the proposed formulas for ballistic applications regarding the underlying assumptions.

## 2 Conventional Calculation of Ballistic Dispersion in Angular Mil

Traditionally, when a projectile exits a gun barrel, the initial downrange velocity $v_{z_0}$ is assumed to be constant, i.e., $\mathrm{Var}[v_{z_0}]=0$, from one round to another. In addition, the drag force is ignored such that the downrange velocity of the projectile at any time remains constant over the entire flight period, i.e., $v_z=v_{z_0}$. In the elevation direction, the initial velocity $v_{y_0}$ is considered to be a zero-mean Gaussian variable, i.e., $E[v_{y_0}]=0$ and $\mathrm{Var}[v_{y_0}]=(\mathrm{SD}[v_{y_0}])^2$. In addition, the effect of gravity is neglected such that the elevation velocity of the projectile at any time remains the same before it hits a target. Lastly, the initial velocity $v_{x_0}$ in the azimuth direction is also considered to be a zero-mean Gaussian variable, i.e., $E[v_{x_0}]=0$ and $\mathrm{Var}[v_{x_0}]=(\mathrm{SD}[v_{x_0}])^2$, and the velocity remains unchanged when no crosswind is taken into account. In short, the net forces are all zeroes throughout the entire flight period in the downrange, elevation, and azimuth directions.

Given a downrange distance of $d_{z_1}$, the flight time over the distance can be written as

$$t_1 = \frac{d_{z_1}}{v_{z_0}} \tag{1}$$

Since $d_{z_1}$ and $v_{z_0}$ are constants, $t_1$ is a constant as well. Over the $t_1$ period of time, the travel distance of the projectile in the azimuth direction can be written as

$$d_{x_1} = v_{x_0} t_1 \tag{2}$$

By taking the expected value on both sides of the equation, the expected value of $d_{x_1}$ can be obtained by

$$E[d_{x_1}] = E[v_{x_0}]t_1 = 0 \tag{3}$$

Similarly, the variance of $d_{x_1}$ can be derived by taking the variance on both sides of the equation as follows:

$$\mathrm{Var}[d_{x_1}] = \mathrm{Var}[v_{x_0}]t_1^2 \tag{4}$$

It can be easily understood that the mean value of $d_{x_1}$ is zero because $v_{x_0}$ is a zero-mean Gaussian variable. The variance of $d_{x_1}$ equals the variance of $v_{x_0}$ multiplied by the square of the flight time. As a result, the dispersion in the azimuth direction at $d_{z_1}$ can be expressed as

$$\mathrm{SD}[d_{x_1}] = \mathrm{SD}[v_{x_0}]t_1 = \mathrm{SD}[v_{x_0}]\frac{d_{z_1}}{v_{z_0}} \tag{5}$$

The ratio $\mathrm{SD}[d_{x_1}]/d_{z_1}$ multiplied by a constant $6400/(2\pi)$ is the conversion of the absolute dispersion to angular mil, adopted by NATO countries. This process can be repeated for another downrange distance $d_{z_2}$. In this case, the flight time can be expressed as $t_2=d_{z_2}/v_{z_0}$. Similarly, $t_2$ is a constant since $d_{z_2}$ and $v_{z_0}$ are both constants. Over the $t_2$ period of time, the travel distance of the projectile in the azimuth direction can be written as

$$d_{x_2} = v_{x_0} t_2 \tag{6}$$

The statistical properties of $d_{x_2}$ can be obtained by taking the expected value and the variance on both sides of the equation as follows:

$$E[d_{x_2}] = E[v_{x_0}]t_2 = 0 \tag{7}$$

and

$$\mathrm{Var}[d_{x_2}] = \mathrm{Var}[v_{x_0}]t_2^2 \tag{8}$$

As a result, the dispersion in the azimuth direction at $d_{z_2}$ can be expressed as

$$\mathrm{SD}[d_{x_2}] = \mathrm{SD}[v_{x_0}]t_2 = \mathrm{SD}[v_{x_0}]\frac{d_{z_2}}{v_{z_0}} \tag{9}$$

Combining Eqs. (5) and (9) yields the following:

$$\frac{\mathrm{SD}[d_{x_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{x_2}]}{d_{z_2}} \tag{10}$$

The ratios of the $x$ dispersion to the corresponding downrange distance are equal. This equation confirms the geometric relationship, as shown in Fig. 1. In the elevation direction, it is also straightforward to derive the relationship as follows:

$$\frac{\mathrm{SD}[d_{y_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{y_2}]}{d_{z_2}} \tag{11}$$

That is, the ratios of the $y$ dispersion to the corresponding downrange distance are equal as well. The aforementioned geometric relationship also holds in the elevation direction. Equations (10) and (11) yield the unit of dispersion in angular mil and have been commonly used to compare dispersions at two or more downrange distances.

## 3 Formulation of Dispersion With a Constant $v_{z_0}$

It has been shown that for high-speed projectiles, the coefficient of variation of $v_{z_0}$ is as small as 0.5% through rigorous studies of 225 distinct barrel shapes and varying in-bore pressure due to the variations in propellant charges [7]. Thus, the assumption that $v_{z_0}=\mathrm{const}$, i.e., $\mathrm{Var}[v_{z_0}]=0$, should be reasonable. This section considers the effects of the external factors: drag force, gravity drop, and crosswind. That is, the flight velocities, $v_z$, $v_y$, and $v_x$, vary over time. In other words, the significance of the three time-varying in-flight velocity components is taken into account in the dispersion formulation. The initial velocities in the azimuth and elevation directions are represented by zero-mean Gaussian random variables, i.e., $E[v_{y_0}]=0$ and $\mathrm{Var}[v_{y_0}]=(\mathrm{SD}[v_{y_0}])^2$, and $E[v_{x_0}]=0$ and $\mathrm{Var}[v_{x_0}]=(\mathrm{SD}[v_{x_0}])^2$, respectively, throughout this study.

**3.1 Downrange Direction.** The downrange velocity of flying projectiles decreases over time because of air drag force. As a result, the ratio of $d_1$ to $t_1$ is no longer equal to that of $d_2$ to $t_2$. It has been shown that the aerodynamic drag force can be written as [8]

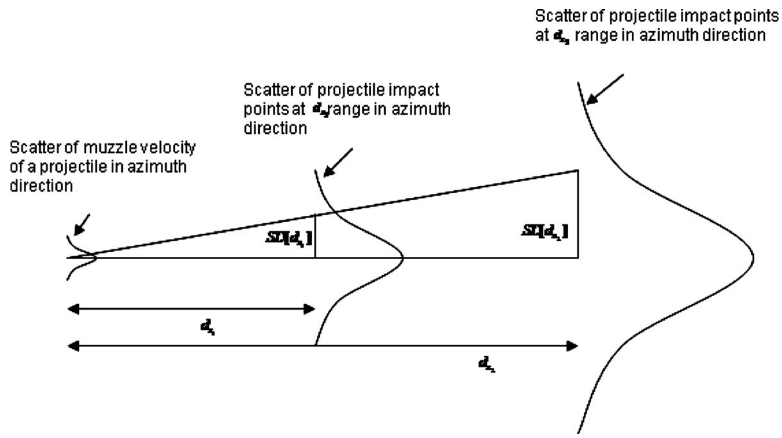


**Fig. 1 Geometric relationship between dispersion and downrange distance**

$$F = -\tfrac{1}{2}\rho A C_d V^2 \tag{12}$$

where $\rho$=air density, $A$=reference area, $C_d$=drag coefficient, and $V^2=v_x^2+v_y^2+v_z^2$=launch velocity of the projectile. Since $v_z \gg v_y$ and $v_z \gg v_x$, the equation in the downrange direction is expressed as

$$\frac{dv_z}{dt} = -C_d^* v_z^2 \tag{13}$$

in which $C_d^*=\rho A C_d/2m$. The parameter $C_d^*$ is a constant for the same ammunition provided that the shape and the mass of the projectile are all alike from one round to another. Subsequently, the velocity can be derived as

$$v_z = \frac{v_{z_0}}{1 + C_d^* v_{z_0} t} \tag{14}$$

As a result, the downrange travel distance can be expressed as

$$d_z = \frac{\ln(1 + C_d^* v_{z_0} t)}{C_d^*} \tag{15}$$

The flight time $t$ can be obtained by rearranging Eq. (15) as

$$t = \frac{1}{C_d^* v_{z_0}}(e^{C_d^* d_z} - 1) \tag{16}$$

The flight time $t$ appears to be nonlinearly proportional to the distance. However, $t$ is not a random variable. For projectiles with a constant $v_{z_0}$, the flight time $t$ is determined given a downrange distance, $d_z$.

**3.2 Elevation Direction.** When a projectile exits a gun tube, it generally possesses an initial velocity in the elevation direction. This velocity is assumed to be a zero-mean Gaussian random variable. The variation in the initial velocity, $v_{y_0}$, from one round to another is considered to cause the spread of the TIP at any range. The scenario that the initial velocity goes downward, i.e., toward the earth, is discussed herein. Based on Eq. (12) and the inclusion of the gravity effect, the equilibrium equation in the elevation direction can be expressed as

$$\frac{dv_y}{dt} = -C_d^* v_z v_y - g \tag{17}$$

The flight velocity of a projectile in this case can be solved as

$$v_y = \frac{v_z}{v_{z_0}}\left(v_{y_0} + \frac{g}{2C_d^* v_{z_0}}(1 - e^{2C_d^* d_z})\right) \tag{18}$$

By taking the expected value and the variance on both sides of the equation, it yields

$$E[v_y] = \frac{v_z}{v_{z_0}}\left(E[v_{y_0}] + \frac{g}{2C_d^* v_{z_0}}(1 - e^{2C_d^* d_z})\right) = \frac{g v_z}{2C_d^* v_{z_0}^2}(1 - e^{2C_d^* d_z}) \tag{19}$$

and

$$\mathrm{SD}[v_y] = \frac{v_z}{v_{z_0}}\mathrm{SD}[v_{y_0}] \tag{20}$$

The expected velocity increase due to gravity is shown in Eq. (19). In addition, the velocity path is a straight line along the range. That is, the ratio of the elevation velocity to the downrange velocity is independent of flight time. Subsequently, the travel distance in the elevation direction can be expressed as

$$d_y = d_z\left(\frac{v_{y_0}}{v_{z_0}} + \frac{g}{2C_d^* v_{z_0}^2}\right) + \quad \frac{g(1 - e^{2C_d^* d_z})}{4C_d^{*2} v_{z_0}^2}\Bigg) \tag{21}$$

Again, by taking the expected value and the variance on both sides of the equation, the statistical properties of $d_y$ can be derived as

$$E[d_y] = d_z g/(2C_d^* v_{z_0}^2) + g(1 - e^{2C_d^* d_z})/(4C_d^{*2} v_{z_0}^2) \tag{22}$$

$$\mathrm{Var}[d_y] = \mathrm{Var}[v_{y_0}]\left(\frac{d_z}{v_{z_0}}\right)^2 \tag{23}$$

and

$$\mathrm{SD}[d_y] = \mathrm{SD}[v_{y_0}]\frac{d_z}{v_{z_0}} \tag{24}$$

The expected projectile drop distance from the aim point due to gravity is shown in Eq. (22). Equation (24) indicates that the dispersion in the elevation direction is the standard deviation of the initial elevation velocity multiplied by the ratio of the downrange travel distance to the initial downrange velocity. In other words, the dispersion is still linearly proportional to the downrange distance. Readers can find that the result here is the same when compared with that in conventional angular dispersion calculation, as shown in Eq. (9). Thus, air drag and gravity are concluded to have no influence on dispersion as $v_{z_0}$ is unchanged from one round to another. Same conclusion can be drawn when the initial velocity is in the opposite direction of the gravity. One can easily derive the equations for the scenario. The relationship between the downrange distance and the expected drop distance due to gravity, along with the distribution of the TIP, is illustrated in Fig. 2. The mean point of impact (MPI) represents the expected impact location for the group of rounds.

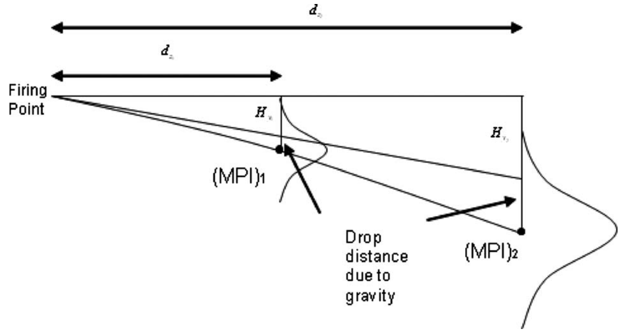


**Fig. 2 Illustration of projectile drop distance due to gravity as the initial velocity going downward**

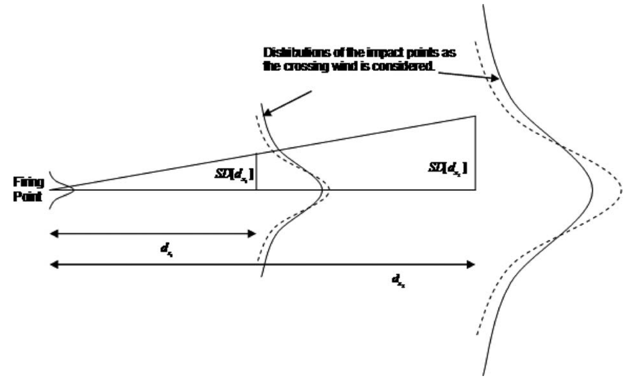


**Fig. 3 Illustration of the increase in the dispersion of target impact points due to crosswind**

**3.3 Azimuth Direction.** When a projectile is fired in a crosswind, its velocity relative to the moving air mass has a small component in the cross-range direction, which causes the projectile to deflect in the direction of the wind. Based on Eq. (12) and the inclusion of crosswind effect, the equilibrium equation in the azimuth direction can be expressed as

$$\frac{dv_x}{dt} = -C_d^* v_z (v_x - w_x) \tag{25}$$

Similarly, the initial velocity of a projectile in the azimuth direction could be in the same or opposite direction of the crosswind. This section discusses the scenario when $w_x$ and $v_x$ in the same direction only. Readers can easily derive the equations as they are in the opposite direction. The flight velocity in the azimuth direction can be expressed as

$$v_x = \frac{v_z}{v_{z_0}} v_{x_0} + w_x \left(1 - \frac{v_z}{v_{z_0}}\right) \tag{26}$$

It can be seen that this equation shows no cross-range velocity due to crosswind initially when $v_{x_0}=0$ and $v_z=v_{z_0}$. As $v_z$ decreases downrange, the wind effect increases even when the wind speed stays the same. In most cases, the wind speed on flying projectiles varies from one round to another. However, when a large number of rounds are being studied, the assumption of a zero-mean Gaussian distribution of the crosswind should be reasonable. That is, the crosswind could be stronger on some rounds and weaker on some other rounds. Nevertheless, due to very short flight periods, the wind speed for each individual round may be considered as a constant for that particular round. By taking the expected value and the variance on both sides of Eq. (26), it yields

$$E[v_x] = E[v_{x_0}] \frac{v_z}{v_{z_0}} + E[w_x]\left(1 - \frac{v_z}{v_{z_0}}\right) = 0 \tag{27}$$

and

$$\mathrm{Var}[v_x] = \mathrm{Var}[v_{x_0}]\left(\frac{v_z}{v_{z_0}}\right)^2 + \mathrm{Var}[w_x]\left(1 - \frac{v_z}{v_{z_0}}\right)^2 + 2\left(1 - \frac{v_z}{v_{z_0}}\right)\frac{v_z}{v_{z_0}}\mathrm{Cov}[v_{x_0}, w_x] \tag{28}$$

When no or low correlation between the initial $x$ velocity and the crosswind speed is expected, the covariance term may be dropped from the equation. The travel distance in the azimuth direction can then be expressed as

$$d_x = \frac{d_z}{v_{z_0}} v_{x_0} + \frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^* v_{z_0}} w_x \tag{29}$$

Similarly, by taking the expected value and the variance on both sides of the equation, it yields

$$E[d_x] = \frac{d_z}{v_{z_0}} E[v_{x_0}] + \frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^* v_{z_0}} E[w_x] = 0 \tag{30}$$

$$\mathrm{Var}[d_x] = \mathrm{Var}[v_{x_0}]\left(\frac{d_z}{v_{z_0}}\right)^2 + \mathrm{Var}[w_x]\left(\frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^* v_{z_0}}\right)^2 + 2\left(\frac{d_z}{v_{z_0}}\right) \times \left(\frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^* v_{z_0}}\right)\mathrm{Cov}[v_{x_0}, w_x] \tag{31}$$

Equation (30) implies that crosswind has no influence on the expected distance deviation. However, it increases the variance by two additional components, as shown in Eq. (31). Since the covariance term is usually small and therefore neglected, the dispersion in the azimuth direction can be written as

$$\mathrm{SD}[d_x] \approx \sqrt{\mathrm{Var}[v_{x_0}]\left(\frac{d_z}{v_{z_0}}\right)^2 + \mathrm{Var}[w_x]\left(\frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^* v_{z_0}}\right)^2} \tag{32}$$

Equivalent to Eq. (29), the travel distance in the azimuth direction at any time $t$ can also be expressed as $d_x = v_{x_0} d_z / v_{z_0} + w_x (t - d_z / v_{z_0})$. When an indoor experiment takes place, i.e., no crosswind is in effect for a certain period, a time shift may be simply appended in the second component of the equation.

Regardless of the relative direction between the initial lateral velocity and the crosswind, the drift distance caused by the crosswind is zero. The mean of $d_x$ is not affected by the crosswind given the assumption that wind speed is normally distributed and has a zero mean across a large number of rounds. In addition, it is certain that the crosswind effect will always yield greater dispersion value, as shown in Eq. (31). Figure 3 illustrates the unchanged MPI at any downrange distance. The new distributions of the impact points due to crosswind are displayed in a solid curve along with the distributions in a dashed curve when crosswind is neglected.

Based on Eq. (32), the dispersions in angular mil at any two downrange distances, $d_{z_1}$ and $d_{z_2}$, can be shown as Eq. (33). In a nutshell, given a certain range, the dispersion in the azimuth direction can be determined by the variance of the initial azimuth velocity of the projectile, the variance of the crosswind velocity, and the initial downrange velocity,

$$\frac{\mathrm{SD}[d_{x_1}]}{\sqrt{\mathrm{Var}[v_{x_0}]\left(\frac{d_{z_1}}{v_{z_0}}\right)^2+\mathrm{Var}[w_x]\left(\frac{e^{C_d^* d_{z_1}}-C_d^* d_{z_1}-1}{C_d^* v_{z_0}}\right)^2}}=\frac{\mathrm{SD}[d_{x_2}]}{\sqrt{\mathrm{Var}[v_{x_0}]\left(\frac{d_{z_2}}{v_{z_0}}\right)^2+\mathrm{Var}[w_x]\left(\frac{e^{C_d^* d_{z_2}}-C_d^* d_{z_2}-1}{C_d^* v_{z_0}}\right)^2}} \tag{33}$$

## 4 Formulation of Dispersion With a Varying $v_{z_0}$

It has been observed that projectile launch velocity in the downrange direction actually deviates from one round to another. This is particularly true for lower speed projectiles that exhibit a higher coefficient of variation from experimental data. As a result, the case that $v_{z_0}$ is considered as a random variable should be addressed. Due to the stochastic nature of $v_{z_0}$, at a certain range, the varying $v_{z_0}$ would lead to uncertain flight time, which in turn affects the outcome of the dispersion in the elevation and azimuth directions. This section intends to formulate the dispersion in a more comprehensive situation.

**4.1 Downrange Direction.** Owing to the variation of $v_{z_0}$, the flight time $t$ at a given distance will be a random variable. As a result, the statistics of $t$ can be obtained by taking the expected value and the variance on both sides of the equation. It yields

$$E[t]=\frac{e^{C_d^* d_z}-1}{C_d^*}E[v_{z_0}^{-1}] \tag{34}$$

and

$$\mathrm{SD}[t]=\frac{e^{C_d^* d_z}-1}{C_d^*}\mathrm{SD}[v_{z_0}^{-1}] \tag{35}$$

The first equation implies that the expected value of the flight time is proportional to the mean of the inverse of the initial downrange velocity. This makes sense since faster projectiles require shorter flight time to reach a target at the same distance. In addition, the variance of the flight time is proportional to the variance of the inverse of the initial downrange velocity.

**4.2 Elevation Direction.** When $v_{z_0}$ is considered a constant, gravity and air drag are found to have no effect on dispersion. As $v_{z_0}$ varies from one round to another, the variation results in a random effect of gravity drop on a projectile at a given range. In other words, it is a completely stochastic process over the projectile trajectory. Again, this section discusses the scenario of initial elevation velocity going downward. The case as the initial velocity going upward is left to readers.

The statistical properties of the $y$ velocity can be obtained from Eq. (18) for this case. Since $v_{y_0}$ and $v_{z_0}$ are both random variables, it yields

$$E[v_y]=v_z\left(E[v_{y_0}v_{z_0}^{-1}]+\frac{g}{2C_d^*}(1-e^{2C_d^* d_z})E[v_{z_0}^{-2}]\right) \tag{36}$$

$$\mathrm{Var}[v_y]=v_z^2\left(\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]+\frac{g^2}{4C_d^{*2}}(1-e^{2C_d^* d_z})^2\mathrm{Var}[v_{z_0}^{-2}]+\frac{g}{C_d^{*2}}(1-e^{2C_d^* d_z})\mathrm{Cov}[v_{y_0}v_{z_0}^{-1},v_{z_0}^{-2}]\right) \tag{37}$$

Due to the variation in the initial downrange velocity, the nonzero expected value of the elevation velocity has an additional component, which is the correlation between the initial elevation velocity and the inverse of the initial downrange velocity. The value may be significant from the physical point of view. Also, based on Eq. (21), the statistics of the travel distance in the elevation direction can be obtained by

$$E[d_y]=d_zE[v_{y_0}v_{z_0}^{-1}]+\frac{g}{2C_d^*}\left(d_z+\frac{1-e^{2C_d^* d_z}}{2C_d^*}\right)E[v_{z_0}^{-2}] \tag{38}$$

$$\mathrm{Var}[d_y]=d_z^2\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]+\frac{g^2}{4C_d^{*2}}\left(d_z+\frac{1-e^{2C_d^* d_z}}{2C_d^*}\right)^2\mathrm{Var}[v_{z_0}^{-2}]+d_z\frac{g}{C_d^*}\left(d_z+\frac{1-e^{2C_d^* d_z}}{2C_d^*}\right)\mathrm{Cov}[v_{y_0}v_{z_0}^{-1},v_{z_0}^{-2}] \tag{39}$$

where

$$\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]=E[(v_{y_0}v_{z_0}^{-1})^2]-(E[v_{y_0}v_{z_0}^{-1}])^2 \tag{40}$$

$$\mathrm{Var}[v_{z_0}^{-2}]=E[v_{z_0}^{-4}]-(E[v_{z_0}^{-2}])^2 \tag{41}$$

and

$$\mathrm{Cov}[v_{y_0}v_{z_0}^{-1},v_{z_0}^{-2}]=E[v_{y_0}v_{z_0}^{-3}]-E[v_{y_0}v_{z_0}^{-1}]E[v_{z_0}^{-2}] \tag{42}$$

It can be observed that due to the variation in $v_{z_0}$, the distribution of the impact points no longer exhibits a zero mean, as indicated in Eq. (38). The expected deviation from the aim point can be determined accordingly. In addition, the quantities $\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]$ and $\mathrm{Var}[v_{z_0}^{-2}]$ are essential for the variance calculations of projectile velocity and travel distance. As mentioned, the covariance component is usually small and, thus, neglected. As a result, the formulation of dispersion can be approximated by

$$\mathrm{SD}[d_y]\approx\sqrt{d_z^2\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]+\frac{g^2}{4C_d^{*2}}\left(d_z+\frac{1-e^{2C_d^* d_z}}{2C_d^*}\right)^2\mathrm{Var}[v_{z_0}^{-2}]} \tag{43}$$

Note that a sign difference may be obtained in the first component of Eq. (38) as the initial velocity going upward, indicating dissimilar expected gravity drop between the cases. Although the sign difference also takes place in the covariance term of Eq. (39), the approximation of dispersion shown in Eq. (43) is identical as the covariance component is neglected.

**4.3 Azimuth Direction.** When the initial azimuth velocity of a projectile is in the same direction of crosswind, the statistics of the flight velocity can be derived from Eq. (26). It yields

$$E[v_x]=v_z(E[v_{x_0}v_{z_0}^{-1}]+E[w_xv_{z_0}^{-1}]) \tag{44}$$

$$\begin{aligned}\mathrm{Var}[v_x]&=v_z^2(\mathrm{Var}[v_{x_0}v_{z_0}^{-1}]+\mathrm{Var}[w_xv_{z_0}^{-1}])+\mathrm{Var}[w_x]\\&\quad-2v_z^2\mathrm{Cov}[v_{x_0}v_{z_0}^{-1},w_xv_{z_0}^{-1}]+2v_z(\mathrm{Cov}[v_{x_0}v_{z_0}^{-1},w_x]\\&\quad-\mathrm{Cov}[w_xv_{z_0}^{-1},w_x])-v_z^2\mathrm{Cov}[v_{x_0}v_{z_0}^{-1},w_x,w_xv_{z_0}^{-1}]\end{aligned} \tag{45}$$

It is shown that due to changing $v_{z_0}$ from one round to another, the expected velocity in the azimuth direction is no longer zero. Because of three variables being considered, the variance of the azimuth velocity is fairly complex. However, for simplicity, one may ignore the covariance terms since the values are usually small compared with the variance quantity. Subsequently, the statistics of the travel distance in the azimuth direction can be obtained from Eq. (29). It yields

$$E[d_x]=d_zE[v_{x_0}v_{z_0}^{-1}]+\frac{e^{C_d^* d_z}-C_d^* d_z-1}{C_d^*}E[w_xv_{z_0}^{-1}] \tag{46}$$

$$\mathrm{Var}[d_x] = d_z^2 \mathrm{Var}[v_{x_0} v_{z_0}^{-1}] + \left( \frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^*} \right)^2 \mathrm{Var}[w_x v_{z_0}^{-1}] + 2 d_z \left( \frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^*} \right) \mathrm{Cov}[v_{x_0} v_{z_0}^{-1}, w_x v_{z_0}^{-1}] \tag{47}$$

where

$$\mathrm{Cov}[v_{x_0} v_{z_0}^{-1}, w_x v_{z_0}^{-1}] = E[v_{x_0} w_x v_{z_0}^{-2}] - E[v_{x_0} v_{z_0}^{-1}] E[w_x v_{z_0}^{-1}] \tag{48}$$

As expected, the nonzero mean of the velocity leads to the nonzero mean of the drift distance, as shown in Eq. (46). When the covariance term of Eq. (47) is neglected, the dispersion can be expressed as

$$\mathrm{SD}[d_x] = \sqrt{d_z^2 \mathrm{Var}[v_{x_0} v_{z_0}^{-1}] + \left( \frac{e^{C_d^* d_z} - C_d^* d_z - 1}{C_d^*} \right)^2 \mathrm{Var}[w_x v_{z_0}^{-1}]} \tag{49}$$

As the initial azimuth velocity and the crosswind are in the opposite directions, the results were found to be similar to those in the elevation direction as discussed previously.

## 5 Correction Factors for Dispersion Comparison at Multiple Ranges

**5.1 Projectile With a Constant $v_{z_0}$ .** It has been shown that with a constant $v_{z_0}$, dispersion in the elevation direction is linearly proportional to the downrange distance even when air drag and gravity are considered. In this case, the geometric relationship shown in Fig. 1 still holds, indicating that the conventional approach to obtaining angular dispersion and making comparisons at multiple ranges is appropriate. However, when crosswind is considered, the dispersion at a longer range tends to be overly estimated in the azimuth direction. Thus, a CF when comparing dispersion at two downrange distances, $d_{z_1}$ and $d_{z_2}$, is proposed as

$$\frac{\mathrm{SD}[d_{x_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{x_2}]}{d_{z_2}} (\mathrm{CF}) \tag{50}$$

Based on Eq. (33), the relationship can be written as

$$\frac{\mathrm{SD}[d_{x_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{x_2}]}{d_{z_2}} \left\{ \frac{\sqrt{\mathrm{Var}[v_{x_0}] \left( \frac{d_{z_1}}{v_{z_0}} \right)^2 + \mathrm{Var}[w_x] \left( \frac{e^{C_d^* d_{z_1}} - C_d^* d_{z_1} - 1}{C_d^* v_{z_0}} \right)^2}}{\sqrt{\mathrm{Var}[v_{x_0}] \left( \frac{d_{z_2}}{v_{z_0}} \right)^2 + \mathrm{Var}[w_x] \left( \frac{e^{C_d^* d_{z_2}} - C_d^* d_{z_2} - 1}{C_d^* v_{z_0}} \right)^2}} \frac{d_{z_2}}{d_{z_1}} \right\} \tag{51}$$

where the components in the curly bracket represent the CF. It can be observed that the CF depends on $\mathrm{Var}[v_{x_0}]$, $\mathrm{Var}[w_x]$, and $C_d^*$. Given any two downrange distances, the CF can be calculated accordingly.

This section adopts a 5.56 mm NATO ammunition for parametric study since it has been under firing test at the Army Research Laboratory over the past several years. The 5.56 mm ammunition is a current U.S. Army standard issue and is manufactured in significantly large quantities. Some of the nominal properties required for dispersion calculation are given in Table 1. In light of the information, the parameter $C_d^*$ is calculated to be 0.001151. The absolute dispersions in the elevation and azimuth directions at 200 m and 500 m ranges are available from previous analysis and are provided in Table 2 [9]. It should be mentioned that the 5.56 mm firing test was conducted in an indoor facility having a length of 300 m where no crosswind was considered.

From a previous analysis, the traditional approach yielded dispersions of 0.2526 and 0.3630 (in angular mil) at 200 m and 500 m, respectively, in the elevation direction. If the initial downrange velocity is assumed to be unchanged from one round to another, no correction is needed. Additionally, the traditional approach yielded dispersions of 0.2949 and 0.3559 (in angular mil) at 200 m and 500 m, respectively, in the azimuth direction. The variances of the initial azimuth velocity and the crosswind speed could be back-calculated by Eq. (32), and they were determined to be 0.053 and 0.23, respectively. As a result, a correction factor of 0.732 is needed such that the comparable dispersion should be 0.2949 and 0.2605, respectively. When a low coefficient of variation of 1% is used for $v_{z_0}$, through Monte Carlo simulation, a correction factor of 0.969 is needed in the elevation direction such that the comparable dispersion should be 0.2526 and 0.3518.

In addition, it is also of interest to assess the degrees of contribution to correction factors by parameters such as $C_d^*$, $\mathrm{Var}[v_{x_0}]$, and $\mathrm{Var}[w_x]$. The $C_d^*$ mainly depends on drag coefficient, bullet mass, and air density. Thus, possible values of each factor are adopted to determine $C_d^*$, as shown in Table 3. The calculated $C_d^*$ values range from 0.001151 to 0.001919. It can be observed that a higher drag coefficient and air density yield higher $C_d^*$. In addition, a lighter projectile mass yields a greater $C_d^*$ value. Overall, the changes in drag coefficient are predominant.

A total of seven cases with different parameter values are adopted and shown in Table 4. The CF at five different ranges, i.e., 200 m, 300 m, 400 m, 500 m, and 600 m, are determined against the baseline dispersion at 100 m. Table 5 gives the CFs for the five

**Table 1 Nominal properties of M855 ammunition**

| | |
|---|---|
| Launch downrange velocity (m/s) | 927 |
| Gravitational acceleration $(\mathrm{m/s^2})$ | 9.81 |
| Bullet reference area $(\mathrm{m^2})$ | $2.55 \times 10^{-5}$ |
| Mass (kg) | 0.004 |
| Air density $(\mathrm{kg/m^3})$ | 1.204 |
| Drag coefficient | 0.3 |

**Table 2 Standard deviation of M855 impact points about MPI at 200 m and 500 m ranges**

| Transverse directions at target location | Standard deviation |
|---|---|
| Elevation distance (m) at 200 m | 0.0579 |
| Elevation distance (m) at 500 m | 0.1747 |
| Azimuth distance (m) at 200 m | 0.0496 |
| Azimuth distance (m) at 500 m | 0.1782 |

**Table 3 List of possible values of $C_d^*$ for high-speed projectiles**

| Case | $C_d$ | $m$ | $\rho$ | $C_d^*$ |
|---|---|---|---|---|
| 1 | 0.4 | 0.004 | 1.204 | 0.001535 |
| 2 | 0.3 | 0.004 | 1.204 | 0.001151 |
| 3 | 0.5 | 0.004 | 1.204 | 0.001919 |
| 4 | 0.4 | 0.0038 | 1.204 | 0.001616 |
| 5 | 0.4 | 0.0042 | 1.204 | 0.001462 |
| 6 | 0.4 | 0.004 | 1.146 | 0.001461 |
| 7 | 0.4 | 0.004 | 1.342 | 0.001711 |

**Table 4 List of parameter values for the case of constant $v_{z_0}$**

| Case | $C_d^*$ | $\mathrm{Var}[v_{x_0}]$ | $\mathrm{Var}[w_x]$ |
|---|---|---|---|
| 1 | 0.001535 | 1 | 1 |
| 2 | 0.001151 | 1 | 1 |
| 3 | 0.001919 | 1 | 1 |
| 4 | 0.001535 | 0.1 | 1 |
| 5 | 0.001535 | 10 | 1 |
| 6 | 0.001535 | 1 | 0.1 |
| 7 | 0.001535 | 0.1 | 10 |

ranges in the azimuth direction. Note that when one is comparing dispersions at 200 m and 500 m, the ratio of the two CFs from the table is the calibration value that should be used.

A lower CF value implies that the nonlinear effects of drag, gravity, and crosswind are more significant. A CF near 1.0 supports the usage of the traditional approach. In Table 5, it can be seen that a higher drag coefficient, a lighter bullet, or dense air each decreases CF. The variance of crosswind speed appears to influence CF significantly, as shown in case 7 where CF deteriorates dramatically. Overall, the correction factors decrease with the ratio of the variances of crosswind to the initial azimuth velocity. When the two variances proportionally increase or decrease, the correction factors remain the same, as seen from comparing cases 5 and 6. Not surprisingly, when the variance of crosswind stays at the same level, the increase in the variance of initial azimuth velocity improves the correction factors. In fact, a CF of 1.0 is anticipated when no crosswind is in effect.

**Table 5 Correction factors against dispersion at 100 m in azimuth direction**

| Case | 200 m | 300 m | 400 m | 500 m | 600 m |
|---|---|---|---|---|---|
| 1 | 0.989 | 0.969 | 0.938 | 0.896 | 0.844 |
| 2 | 0.994 | 0.983 | 0.967 | 0.945 | 0.917 |
| 3 | 0.982 | 0.948 | 0.898 | 0.832 | 0.755 |
| 4 | 0.909 | 0.785 | 0.659 | 0.549 | 0.456 |
| 5 | 0.999 | 0.997 | 0.993 | 0.988 | 0.980 |
| 6 | 0.999 | 0.997 | 0.993 | 0.988 | 0.980 |
| 7 | 0.651 | 0.446 | 0.327 | 0.250 | 0.198 |

**5.2 Projectile With a Varying $v_{z_0}$.** It has been shown that with a varying $v_{z_0}$, the angular dispersion at a longer range tends to be overestimated in both elevation and azimuth directions. Therefore, CF will be formulated in both cases. In the elevation direction, one can write a comparison equation such that

$$\frac{\mathrm{SD}[d_{y_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{y_2}]}{d_{z_2}}(\mathrm{CF}) \tag{52}$$

Based on Eq. (43), given two different downrange distances, the relationship can be written as

$$\frac{\mathrm{SD}[d_{y_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{y_2}]}{d_{z_2}}\left\{\frac{\sqrt{d_{z_1}^2\mathrm{Var}[v_{y_0}v_{z_0}^{-1}] + \dfrac{g^2}{4C_d^{*2}}\left(d_{z_1} + \dfrac{1-e^{2C_d^* d_{z_1}}}{2C_d^*}\right)^2 \mathrm{Var}[v_{z_0}^{-2}]}}{\sqrt{d_{z_2}^2\mathrm{Var}[v_{y_0}v_{z_0}^{-1}] + \dfrac{g^2}{4C_d^{*2}}\left(d_{z_2} + \dfrac{1-e^{2C_d^* d_{z_2}}}{2C_d^*}\right)^2 \mathrm{Var}[v_{z_0}^{-2}]}}\frac{d_{z_2}}{d_{z_1}}\right\} \tag{53}$$

The expression in the curly bracket represents the CF. It can be seen that the quantities $\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]$ and $\mathrm{Var}[v_{z_0}^{-2}]$ need to be calculated in order to determine the CF. Similarly, in the azimuth direction, based on Eq. (49), the relationship at two different downrange distances can be obtained by

$$\frac{\mathrm{SD}[d_{x_1}]}{d_{z_1}} = \frac{\mathrm{SD}[d_{x_2}]}{d_{z_2}}\left\{\frac{\sqrt{d_{z_1}^2\mathrm{Var}[v_{x_0}v_{z_0}^{-1}] + \left(\dfrac{e^{C_d^* d_{z_1}} - C_d^* d_{z_1} - 1}{C_d^*}\right)^2 \mathrm{Var}[w_x v_{z_0}^{-1}]}}{\sqrt{d_{z_2}^2\mathrm{Var}[v_{x_0}v_{z_0}^{-1}] + \left(\dfrac{e^{C_d^* d_{z_2}} - C_d^* d_{z_2} - 1}{C_d^*}\right)^2 \mathrm{Var}[w_x v_{z_0}^{-1}]}}\frac{d_{z_2}}{d_{z_1}}\right\} \tag{54}$$

The expression in the curly brackets represents the CF. It can be seen that the quantities $\mathrm{Var}[v_{x_0}v_{z_0}^{-1}]$ and $\mathrm{Var}[w_x v_{z_0}^{-1}]$ need to be calculated in order to determine the CF. Overall, the CF depends on the correlation of the inverse of the initial downrange velocity with the initial azimuth velocity and with the crosswind. Equations (53) and (54) require the information of $v_{x_0}$, $v_{y_0}$, $v_{z_0}$, and $w_x$. In order to better estimate their correlations, a good number of experimental data are necessary. The tediousness and complexity of data collection for all three velocity components and time-dependent wind speeds for a sizable number of shots could hinder using the formulation. In some cases, estimation from statistical inference may be necessary. In other cases, historical data for a weapon system could be utilized to compensate the difficulty.

No complete experimental data for low-speed projectiles are available for demonstration here. On one hand, the experiment that was designed to capture information required for the CF calculations was very limited. Much experimental data were insufficient to provide a thorough study about this matter. On the other hand, significant portions of the projectiles in this category are the ammunition in which the dispersion data are fairly sensitive. The

Table 6 List of possible values of $C_d^*$ for low-speed projectiles

| Combination | $C_d$ | $m$ | $\rho$ | $C_d^*$ |
|---|---|---|---|---|
| 1 | 2.7 | 0.179 | 1.204 | 0.01178 |
| 2 | 2.1 | 0.179 | 1.204 | 0.00916 |
| 3 | 3.3 | 0.179 | 1.204 | 0.01440 |
| 4 | 2.7 | 0.174 | 1.204 | 0.01212 |
| 5 | 2.7 | 0.184 | 1.204 | 0.01146 |
| 6 | 2.7 | 0.179 | 1.146 | 0.01121 |
| 7 | 2.7 | 0.179 | 1.342 | 0.01313 |

inclusion of such data could escalate the permission issue for public release. Thus, readers are left to utilize Eqs. (53) and (54) for their practices. Nevertheless, it should be straightforward to follow the description as outlined for high-speed projectile application. Generally, it is difficult to directly derive the quantities of $\mathrm{Var}[v_{y_0}v_{z_0}^{-1}]$, $\mathrm{Var}[v_{z_0}^{-2}]$, $\mathrm{Var}[v_{x_0}v_{z_0}^{-1}]$, and $\mathrm{Var}[w_x v_{z_0}^{-1}]$. Instead, one can collect some data from $v_{x_0}$, $v_{y_0}$, $v_{z_0}$, and $w_x$ to estimate their individual statistical properties. By leveraging the assumption of Gaussian distribution, these quantities can be obtained through Monte Carlo simulations.

Alternatively, a notional low-speed projectile is adopted for the sensitivity study. It possesses a mass of 0.179 kg, a reference area of 0.0013 m$^2$, and a drag coefficient of 2.7. The average initial downrange velocity of the projectile is 74.3 m/s, with a standard deviation of 0.794 m/s. The values of drag coefficient, projectile mass, and air density are perturbed to determine a possible range of $C_d^*$. The calculated $C_d^*$ values are given in Table 6. It can be seen that the drag coefficient is the biggest driver that results in maximum and minimum $C_d^*$ values of 0.01440 and 0.00916, respectively. The extreme values, along with the average $C_d^*$, are used for dispersion study.

Based on Eqs. (53) and (54), the calculation of CF requires the quantity of $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{y_0}]$, $\mathrm{Var}[\nu_{z_0}^{-2}]$, $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{x_0}]$, and $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{w}]$, which are computed to be $6.79\times10^{-3}$, $1.49\times10^{-11}$, $1.70\times10^{-3}$, and $3.47\times10^{-3}$, respectively. A wind speed with a standard deviation of 4.5 m/s is utilized. Since no closed-form solution is available for the variances, these values are obtained through Monte Carlo simulation. A total of seven cases with different parameter values are created and given in Table 7. The first three cases are to assess the effect of $C_d^*$ on CF. The next two cases are to investigate the influence of $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{y_0}]$ and $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{x_0}]$ on CF, and the last two cases are to study the impact of $\mathrm{Var}[\nu_{z_0}^{-2}]$ and $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{w}]$.

The CFs at five different ranges, i.e., 20 m, 30 m, 40 m, 50 m, and 60 m, are calculated against the baseline dispersion at 10 m. The calculated CFs for the elevation and azimuth directions are provided in Tables 8 and 9, respectively. It was found that correction on angular dispersion in the elevation direction may not be needed in many cases. The changes in $C_d^*$ have little effect on the CF. The primary driver is the variation in the initial downrange, as shown in case 6. The larger the variance, the smaller the CF. As discussed previously, a CF of 1.0 was obtained when $v_{z_0}$ is constant. In the azimuth direction, a lower value of $C_d^*$ improves CF. At the range of 60 m, a high drag coefficient may result in a CF as low as 0.77. Overall, the increase in the variation of the initial azimuth velocity has a limited impact on CF. The major driver is the variance of the product of the crosswind velocity and the inverse of the initial downrange velocity. One can observe from cases 5 and 6 that when $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{w_x}]$ is substantially less than $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{x_0}]$, the CF may deteriorate to a surprisingly low level of 0.203. Since the increase in the variation in initial downrange velocity magnifies both variances of the products simultaneously, a marginal impact on CF is expected. As a result, the variation in crosswind velocity appears to be dominant, which is in line with the conclusion, as described in the application of high-speed projectiles.

Table 7 List of parameter values for the case of varying $v_{z_0}$

| Case | $C_d^*$ | $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{y_0}]$ | $\mathrm{Var}[\nu_{z_0}^{-2}]$ | $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{x_0}]$ | $\mathrm{Var}[\nu_{z_0}^{-1}\nu_{w_x}]$ |
|---|---|---|---|---|---|
| 1 | 0.01178 | $6.79\times10^{-3}$ | $1.49\times10^{-11}$ | $1.70\times10^{-3}$ | $3.47\times10^{-3}$ |
| 2 | 0.00916 | $6.79\times10^{-3}$ | $1.49\times10^{-11}$ | $1.70\times10^{-3}$ | $3.47\times10^{-3}$ |
| 3 | 0.01440 | $6.79\times10^{-3}$ | $1.49\times10^{-11}$ | $1.70\times10^{-3}$ | $3.47\times10^{-3}$ |
| 4 | 0.01178 | $6.79\times10^{-2}$ | $1.49\times10^{-11}$ | $1.70\times10^{-2}$ | $3.47\times10^{-3}$ |
| 5 | 0.01178 | $6.79\times10^{-4}$ | $1.49\times10^{-11}$ | $1.70\times10^{-4}$ | $3.47\times10^{-3}$ |
| 6 | 0.01178 | $6.79\times10^{-4}$ | $1.49\times10^{-10}$ | $1.70\times10^{-4}$ | $3.47\times10^{-2}$ |
| 7 | 0.01178 | $6.79\times10^{-2}$ | $1.49\times10^{-12}$ | $1.70\times10^{-2}$ | $3.47\times10^{-4}$ |

Table 8 Correction factors against dispersion at 10 m in elevation direction

| Case | 20 m | 30 m | 40 m | 50 m | 60 m |
|---|---|---|---|---|---|
| 1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 2 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 3 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 4 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 5 | 1.000 | 1.000 | 0.999 | 0.998 | 0.997 |
| 6 | 0.999 | 0.996 | 0.992 | 0.985 | 0.974 |
| 7 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

Table 9 Correction factors against dispersion at 10 m in azimuth direction

| Case | 20 m | 30 m | 40 m | 50 m | 60 m |
|---|---|---|---|---|---|
| 1 | 0.988 | 0.965 | 0.933 | 0.892 | 0.842 |
| 2 | 0.993 | 0.980 | 0.961 | 0.936 | 0.905 |
| 3 | 0.981 | 0.947 | 0.898 | 0.838 | 0.770 |
| 4 | 0.999 | 0.996 | 0.993 | 0.987 | 0.980 |
| 5 | 0.899 | 0.771 | 0.647 | 0.541 | 0.455 |
| 6 | 0.639 | 0.440 | 0.325 | 0.252 | 0.203 |
| 7 | 1.000 | 1.000 | 0.999 | 0.999 | 0.998 |

## 6 Conclusion

This study formulated ballistic dispersion based on the assumption that the spread of TIPs is primarily driven by the variation in the initial velocity of projectiles as they exit a gun barrel. In this study, the variation in the initial velocity was considered to be a Gaussian distribution in both elevation and azimuth directions. The formulation started with a simplistic case, i.e., an ideal situation where no external forces are imparted to the projectiles over the entire flight period. This scenario agrees with the conventional calculation of ballistic dispersion in angular mil. The angular dispersion was obtained from dividing the standard deviation of TIP by the corresponding downrange distance. The dimensionless unit enables us to make dispersion comparisons at multiple downrange distances. Simply speaking, the comparison was based on a linear relationship between the absolute dispersion and the downrange distance, and the relationship was illustrated.

The formulation of dispersion was extended to a more realistic situation where air drag, gravity drop, and crosswind were taken into account. Projectiles, of which the initial downrange velocity may be considered as a constant from one round to another, were first investigated. In the elevation direction, dispersion was found to be linearly proportional to the downrange distance, whose result agrees with the ideal case. In other words, air drag and gravity make no contribution to dispersion. In the azimuth direction, the drift distance is expected to be zero. However, dispersion is always magnified because of crosswind regardless of the air flow direction. The larger the variance of crosswind velocity, the greater the dispersion.

Furthermore, dispersion was formulated for projectiles that possess varying initial downrange velocity from one round to another, which is even closer to reality. The variation in the velocity was modeled by a Gaussian random variable, the same representation as the initial elevation and azimuth velocities. In this case, the flight time to a given downrange distance is uncertain, owing to the stochastic property of the projectile launch velocity. As a result, the variation, in conjunction with gravity effect, becomes a significant factor in determining dispersion in the elevation direction. In the azimuth direction, three random variables were involved in the dispersion formulation. Thus, a complex expression was derived, which requires the computation of the correlations of the initial downrange velocity with the initial azimuth velocity and with the crosswind velocity. While sizable amounts of data to determine the correlations may not be feasible, the statistics of each individual factor can be estimated with a few data points, followed by comprehensive Monte Carlo simulation for the correlation calculation.

In this study, variant physical conditions were utilized to formulate general ballistic dispersion from a stochastic approach. The theoretical formulation, which links initial projectile velocities to target impact distribution, may be useful to facilitate the assessment of weapon performance. For instance, with existing firing data, one can estimate the initial elevation and azimuth velocities for the weapon system. On the other hand, when the projectile exit conditions at the muzzle are specified with a weapon/ammunition system, it would be straightforward to determine dispersion outcome in both elevation and azimuth directions.

The derived formulations confirm a highly nonlinear relationship of dispersion with the downrange distance in several cases. While the traditional conversion to dispersion in angular mil is quick and easy, the quantity tends to be overestimated, particularly at longer distances. Thus, CFs are proposed for dispersion comparison at multiple downrange distances. Based on a parametric study, CFs were found to be nontrivial in several cases. In the elevation direction, the variation in initial downrange velocity was shown to be the primary contributor to the CF. The effects of drag coefficient, projectile mass, and air density were very limited. In the azimuth direction, the variation in crosswind velocity was determined to be the major driver for the CF. The drag coefficient, projectile mass, and air density were found to slightly influence the CF. Through this quantitative study, one can better understand the degree of impact on ballistic dispersion due to variant projectile conditions at the muzzle and external force fields.

## Acknowledgment

The author wishes to acknowledge valuable comments offered by Dr. Paul Weinacht throughout the preparation of this study.

This work was supported in part by the U.S. Army Project Manager, Maneuver Ammunition Systems.

## Nomenclature

$x$ = azimuth direction
$y$ = elevation direction
$z$ = downrange direction
$x_0$ = gun muzzle position at firing in the azimuth direction
$y_0$ = gun muzzle position at firing in the elevation direction
$z_0$ = gun muzzle position at firing in the downrange direction
$\bar{x}_p$ = MPI position in the azimuth direction
$\bar{y}_p$ = MPI position in the elevation direction
$\bar{z}_p$ = MPI position in the downrange direction
$d_x$ = separation distance of an impact point from MPI in the azimuth direction
$d_y$ = separation distance of an impact point from MPI in the elevation direction
$d_z$ = downrange distance from the firing position $(x_0, y_0, z_0)$
$H_y$ = separation distance between the firing point and the aim point in the elevation direction
$v_{x_0}$ = initial velocity of a projectile in the azimuth direction at a gun muzzle
$v_{y_0}$ = initial velocity of a projectile in the elevation direction at a gun muzzle
$v_{z_0}$ = initial velocity of a projectile in the downrange direction at a gun muzzle
$v_x$ = flight velocity of a projectile in the azimuth direction
$v_y$ = flight velocity of a projectile in the elevation direction
$v_z$ = flight velocity of a projectile in the downrange direction
$w_x$ = wind velocity in the azimuth direction
$t$ = flight time
$g$ = gravitational acceleration
$E[\Omega]$ = expected value of a random variable $\Omega$
$\mathrm{Var}[\Omega]$ = variance of a random variable $\Omega$
$\mathrm{SD}[\Omega]$ = standard deviation of a random variable $\Omega$
$\mathrm{Cov}[\Omega_1, \Omega_2]$ = covariance between two random variables $\Omega_1$ and $\Omega_2$

## References

[1] Weinacht, P., Newill, J., and Conroy, P., 2005, "Conceptual Design Approach for Small-Caliber Aeroballistics With Application to 5.56-mm Ammunition," U.S. Army Research Laboratory, Report No. ARL-TR-3620.
[2] Soencksen, K., Newill, J., and Plostins, P., 2000, "Aerodynamics of the 120-mm M831A1 Projectile: Analysis of Free Flight Experimental Data," U.S. Army Research Laboratory, Report No. ARL-TR-2307.
[3] Erline, T., and Hathaway, M., 1999, "Dispersion Analysis of the XM881 Armor-Piercing, Fin-Stabilized, Discarding Sabot Projectile," U.S. Army Research Laboratory, Report No. ARL-MR-433.
[4] Zielinski, A., Weinacht, P., Webb, D., and Soencksen, K., 1997, "An Investigation of the Ballistic Performance for an Electromagnetic Gun-Launched Projectile," U.S. Army Research Laboratory, Report No. ARL-TR-1361.
[5] Jankovic, S., Gallant, J., and Celens, E., 1999, "Dispersion of an Artillery Projectile Due to Unbalance," *Proceedings of the 18th International Symposium on Ballistics*, San Antonio, TX.
[6] Wilkerson, S., 1995, "Possible Effects on Gun Tube Straightness on Dispersion," U.S. Army Research Laboratory, Report No. ARL-TR-767.
[7] Chen, M., 2007, "Design of Experiments Study of the Initial Extended Area Protection and Survivability Projectile System," U.S. Army Research Laboratory, Report No. ARL-TR-4109.
[8] Carlucci, D., and Jacobson, S., 2008, *Ballistics: Theory and Design of Guns and Ammunition*, CRC, Boca Raton, FL.
[9] Chen, M., Keppinger, R., Celmins, I., and Guidos, B., 2009, "Study of Dispersion of 5.56-mm Weapon Systems," *Proceedings of the 80th Shock and Vibration Symposium*, San Diego, CA.

| NO. OF COPIES | ORGANIZATION |
|---|---|
| 1 (PDF) | DEFENSE TECHNICAL INFORMATION CTR<br>DTIC OCA<br>8725 JOHN J KINGMAN RD<br>STE 0944<br>FORT BELVOIR VA 22060-6218 |
| 1 (PDF) | DIRECTOR<br>US ARMY RESEARCH LAB<br>RDRL CIO LL<br>2800 POWDER MILL RD<br>ADELPHI MD 20783-1197 |
| 1 (PDF) | GOVT PRINTG OFC<br>A MALHOTRA<br>732 N CAPITOL ST NW<br>WASHINGTON DC 20401 |
| 1 (PDF) | RDRL WML G<br>M CHEN |